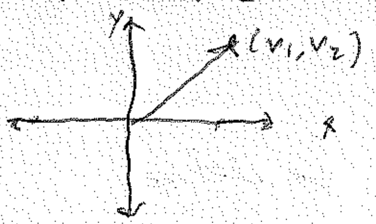
Chapter 1 Vectors, matrices, and applications

Sec 1.1 Vectors

- A <u>vector</u> is an ordered pair in $\mathbb{R}^2$ $\langle 1,2 \rangle$
  " " triple in $\mathbb{R}^3$ $\langle 1,2,3 \rangle$
  " " n-tuple in $\mathbb{R}^n$ $\langle v_1, \ldots, v_n \rangle$

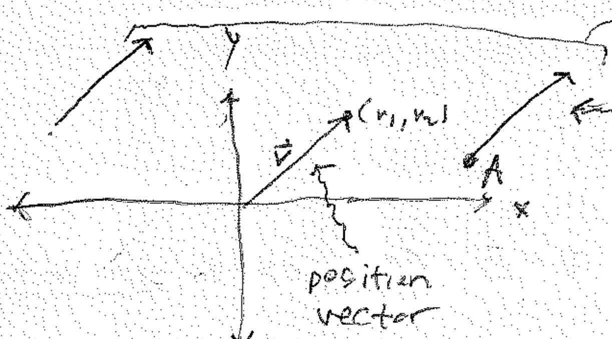
- <u>Notation</u>:
  - $\vec{v} = \mathbf{v} = \langle v_1, v_2 \rangle$
  - $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = [v_1 \ v_2]^T$
  - In $\mathbb{R}^3$, $\vec{v} = \langle v_1, v_2, v_3 \rangle = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$

- $\vec{v} = \langle v_1, v_2 \rangle$ can be visualized as:



a <u>position vector</u> = directed line segment from the origin to (v_1, v_2).

- For convenience, we'd like to move vectors around:



all the "same" vector

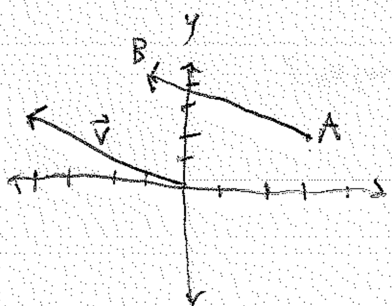
"representative" of $\vec{v}$ that starts at the point A.

- $\overrightarrow{AB}$ is the vector from point A to point B.

<u>Example</u>: $A = (3,2)$, $B = (-1,4)$

$$\vec{v} = \langle b_1 - a_1, b_2 - a_2 \rangle = \langle -1-3, 4-2 \rangle = \langle -4, 2 \rangle$$

is a representative for $\overrightarrow{AB}$



- <u>Length</u> of a vector (or <u>magnitude</u> or <u>norm</u>) will be denoted by $\|\vec{v}\|$ or $|\vec{v}|$ or $d(A,B)$

- For $\vec{v} = \langle v_1, v_2, v_3 \rangle = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle$ we define

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}$$

and analogously for $\vec{v} = \langle v_1, \dots, v_n \rangle$.

<u>Example</u> $\vec{v} = \langle -2, 2 \rangle$

$$\|\vec{v}\| = \sqrt{(-2)^2 + (2)^2} = \sqrt{8}$$

<u>Notice</u>
- $\|\vec{v}\| = 0$ if and only if $\vec{v} = \vec{0} = \langle 0, \dots, 0 \rangle$ = the <u>zero vector</u>.
- $\|\alpha \vec{v}\| = |\alpha| \, \|\vec{v}\|$ $\alpha > 0$: same direction as $\vec{v}$
  $\alpha < 0$: opposite direction

- If $\vec{v}$ has length 1, $\|\vec{v}\| = 1$, $\vec{v}$ is called a <u>unit vector</u>.
- In $\mathbb{R}^3$, $\vec{i} = \langle 1,0,0 \rangle$, $\vec{j} = \langle 0,1,0 \rangle$, $\vec{k} = \langle 0,0,1 \rangle$ are the <u>standard basis vectors</u>.

<u>Example</u> Find the unit vector in the direction of $\vec{v}$

<u>Example</u> Find the unit vector in the direction of

$$\vec{v} = \langle 1, -2, 1 \rangle = \langle 1, 0, 0 \rangle - 2\langle 0, 1, 0 \rangle + \langle 0, 1, 0 \rangle$$
$$= \vec{i} \quad - 2\vec{j} \quad + \vec{k}$$

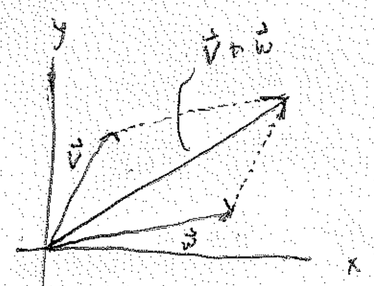
<u>Soln</u> Divide $\vec{v}$ by its norm (<u>normalize</u> $\vec{v}$)

$\|\vec{v}\| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}$, so the vector we want is

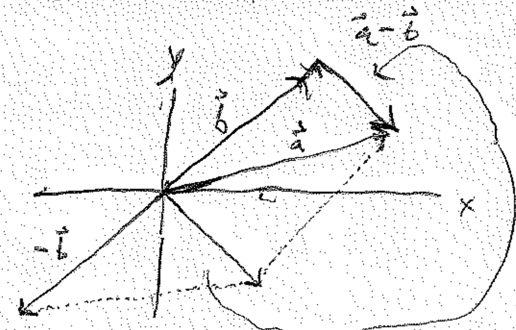
$$\frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{6}}(\vec{i} - 2\vec{j} + \vec{k}).$$

- <u>Vector addition / subtraction</u>: $\vec{v} = \langle a_1, b_1 \rangle$, $\vec{w} = \langle a_2, b_2 \rangle$
  $\vec{v} \pm \vec{w} = \langle a_1 \pm a_2, b_1 \pm b_2 \rangle$

- Addition:



- Subtraction: think $\vec{a} + (-\vec{b})$



starts at end of $\vec{b}$
ends at end of $\vec{a}$

<u>Sec 1.3</u> The dot product

- Let $\vec{v} = \langle v_1, v_2, v_3 \rangle$, $\vec{w} = \langle w_1, w_2, w_3 \rangle$. The <u>dot product</u> (or scalar product or inner product) of $\vec{v}$ and $\vec{w}$ is a real number given by

  $\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$

- Notice that $\vec{v} \cdot \vec{w} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$.

- Notice that $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$, so $\sqrt{\vec{v} \cdot \vec{v}} = \|\vec{v}\|$.

<u>Example</u> $\langle 0, 3, -7 \rangle \cdot \langle 2, 3, 1 \rangle = 0(2) + 3(3) + (-7)(1) = 2$.

- Other properties: $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ (distributivity)
  $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$ (commutativity)
  $(c\vec{v}) \cdot \vec{w} = \vec{v} \cdot (c\vec{w}) = c(\vec{v} \cdot \vec{w})$

- Geometric version of the dot product

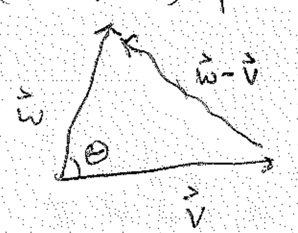
The angle θ between $\vec{v}$ and $\vec{w}$ is understood to be the smaller of the two angles, so $0 \leq \theta \leq \pi$

If $\vec{v}$ and $\vec{w}$ are parallel, then: $\theta = 0$ if they have the same direction
$\theta = \pi$ if they have opposite directions.

- If $\vec{v}$ and $\vec{w}$ are vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, then

$\boxed{\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos \theta}$, or equivalently, $\boxed{\cos\theta = \dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|}}$

(Partial) proof: Use the law of cosines



$$\underbrace{\|v-w\|^2}\ = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\|\vec{w}\|\cos\theta$$

LHS is $(\vec{v}-\vec{w})\cdot(\vec{v}-\vec{w}) = \vec{v}\cdot\vec{v} - 2(\vec{v}\cdot\vec{w}) + \vec{w}\cdot\vec{w}$
$= \|\vec{v}\|^2 - 2(\vec{v}\cdot\vec{w}) + \|\vec{w}\|^2$

So we have $\cancel{\|\vec{v}\|^2} - 2(\vec{v}\cdot\vec{w}) + \cancel{\|\vec{w}\|^2} = \cancel{\|\vec{v}\|^2} + \cancel{\|\vec{w}\|^2} - 2\|\vec{v}\|\|\vec{w}\|\cos\theta$

$\longrightarrow \vec{v}\cdot\vec{w} = \|\vec{v}\|\|\vec{w}\|\cos\theta$.

- $\vec{v}$ and $\vec{w}$ are <u>orthogonal/perpendicular</u> if the angle between them is $\frac{\pi}{2}$ rad $= 90^\circ$.

- <u>Test for orthogonality</u>: Let $\vec{v}$ and $\vec{w}$ be two nonzero vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$.

Then $\vec{v}$ and $\vec{w}$ are orthogonal if and only if $\vec{v}\cdot\vec{w} = 0$.

proof: If $\theta = \frac{\pi}{2}$, then $\|\vec{v}\|\|\vec{w}\|\cos\frac{\pi}{2} = \|\vec{v}\|\|\vec{w}\|(0) = \vec{v}\cdot\vec{w} = 0$.

If $\vec{v}\cdot\vec{w} = 0$, then $\|\vec{v}\|\|\vec{w}\|\cos\theta = 0$,
so $\cos\theta = 0$ (since $\vec{v}$ and $\vec{w}$ are nonzero vectors)
$\longrightarrow \theta = \frac{\pi}{2}$ (since $0 \leq \theta \leq \pi$).

- Similarly, $\vec{v}$ and $\vec{w}$ are parallel ($\theta = 0$ or π) if and only if $\vec{v}\cdot\vec{w} = \pm\|\vec{v}\|\|\vec{w}\|$.

<u>Example</u> Are $\vec{v} = 4\vec{i} - \vec{j} + \vec{k}$ and $\vec{w} = \vec{j} + 3\vec{k}$ orthogonal, parallel, or neither?

<u>Soln</u> $\vec{v}\cdot\vec{w} = 4(0) + (-1)(1) + 1(3) = 2 \neq 0 \rightarrow$ not orthogonal

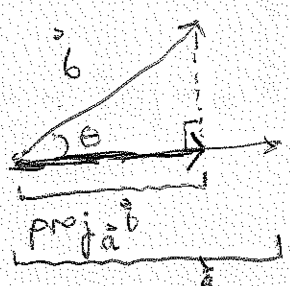
$\|\vec{v}\| = \sqrt{4^2 + (-1)^2 + 1^2} = \sqrt{18}$

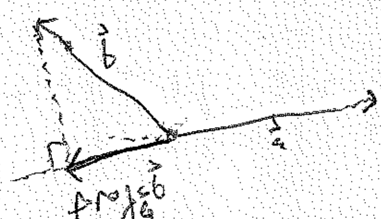
$\|\vec{w}\| = \sqrt{0^2 + 1^2 + 3^2} = \sqrt{10}$

$\vec{v}\cdot\vec{w} = 2 \neq \pm\sqrt{18}\cdot\sqrt{10} = \pm\|\vec{v}\|\,\|\vec{w}\| \longrightarrow$ not parallel

$\longrightarrow \boxed{\text{neither}}$. In fact, $\cos\theta = \dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\,\|\vec{w}\|}$, so the angle between $\vec{v}$ and $\vec{w}$ is $\arccos\left(\dfrac{2}{\sqrt{18}\sqrt{10}}\right) \approx 1.42$ rad.

- <u>Projections</u>
- The (orthogonal) projection of $\vec{b}$ onto $\vec{a}$ is defined as $\boxed{\operatorname{proj}_{\vec{a}}\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|^2}\vec{a}}$.





Drop down from end of $\vec{b}$ until you hit the line parallel to $\vec{a}$, forming a right angle.

- Geometrically, $\operatorname{proj}_{\vec{a}}\vec{b}$ is the vector that is:
  - parallel to $\vec{a}$
  - starts at the common initial point of $\vec{a}$ and $\vec{b}$.
  - ends where the dashed line above hits the line parallel to $\vec{a}$.
- Given this geometric interpretation, why does $\operatorname{proj}_{\vec{a}}\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|^2}\vec{a}$?

With θ as above, $\cos\theta = \dfrac{adj}{hyp} = \dfrac{\|\operatorname{proj}_{\vec{a}}\vec{b}\|}{\|\vec{b}\|}$

$\longrightarrow \|\operatorname{proj}_{\vec{a}}\vec{b}\| = \|\vec{b}\|\cos\theta$

$\longrightarrow \|\operatorname{proj}_{\vec{a}}\vec{b} = \|\vec{b}\|\dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|} = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|}$.

Now we have the length, which we multiply by the unit vector $\frac{\vec{a}}{\|\vec{a}\|}$ in the direction of $\vec{a}$:

$$\longrightarrow \text{proj}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|}\left(\frac{\vec{a}}{\|\vec{a}\|}\right) = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|^2}\vec{a}.$$

- It's norm $\|\text{proj}_{\vec{a}}\vec{b}\|$ is called the scalar projection of $\vec{b}$ onto $\vec{a}$.

Sec 1.5 The cross product

- The cross product of $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ and $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$ is the vector $\vec{v}\times\vec{w}$ "$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$" $= \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}\vec{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}\vec{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}\vec{k}$.

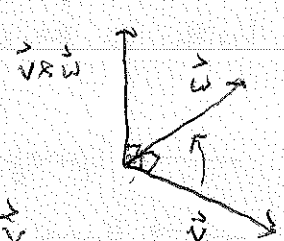
Example $\vec{v} = \vec{i} + 2\vec{j} + 3\vec{k}$ and $w = 2\vec{i} + 4\vec{j} + \vec{k}$

$$\vec{v}\times\vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 2 & 4 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}\vec{k} = -10\vec{i} - 5\vec{j}.$$

- Properties:

$$\vec{v}\times\vec{w} = -(\vec{w}\times\vec{v})$$
$$\vec{v}\times\vec{v} = \vec{0}$$
$$\vec{u}\times(\vec{v}+\vec{w}) = \vec{u}\times\vec{v} + \vec{u}\times\vec{w}$$
$$(\vec{u}+\vec{v})\times\vec{w} = \vec{u}\times\vec{w} + \vec{v}\times\vec{w}$$
$$c(\vec{v}\times\vec{w}) = (c\vec{v})\times\vec{w} = \vec{v}\times(c\vec{w})$$
$$u\cdot(\vec{v}\times\vec{w}) = (\vec{u}\times\vec{v})\cdot\vec{w}$$

- Interesting properties:

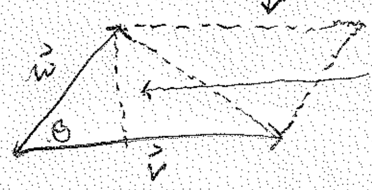


- $\vec{v}\times\vec{w}$ is orthogonal to $\vec{v}$ and $\vec{w}$
- $\|\vec{v}\times\vec{w}\| = \|\vec{v}\|\|\vec{w}\|\sin\theta$
- Right hand rule: place right hand in direction of $\vec{v}$
  curl your fingers from $\vec{v}$ to $\vec{w}$
  your thumb points in the direction of $\vec{v}\times\vec{w}$

- Test for parallel vectors: nonzero vectors $\vec{v}$ and $\vec{w}$ are parallel if and only if $\vec{v} \times \vec{w} = \vec{0}$ (Why?)

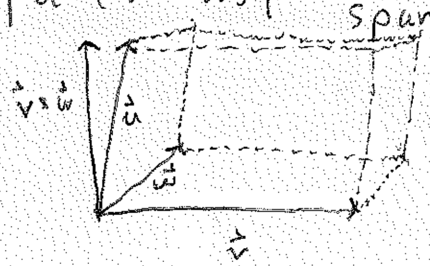
- $\|\vec{v} \times \vec{w}\|$ = area of parallelogram spanned by $\vec{v}$ and $\vec{w}$

(so $\frac{1}{2}\|\vec{v} \times \vec{w}\|$ = area of the triangle below)



- $\vec{u} \cdot (\vec{v} \times \vec{w})$ is called a scalar triple product, and

$|u \cdot (\vec{v} \times \vec{w})|$ = volume of parallelopiped spanned by $\vec{u}, \vec{v}$, and $\vec{w}$ $= \text{abs}\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$



- Intuition for dot and cross product:

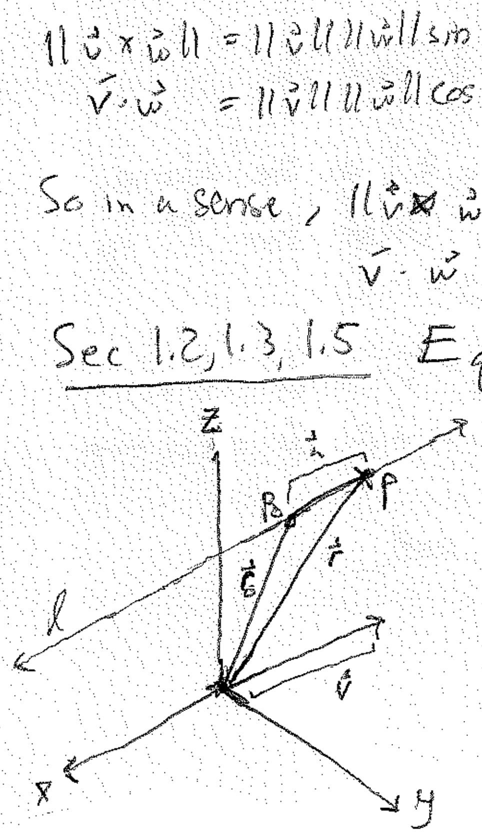
$\|\vec{v} \times \vec{w}\| = \|\vec{v}\|\|\vec{w}\| \sin\theta$

$\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos\theta$

Unit vectors are:	parallel		perpendicular
if $\|$cross product$\|$ is	0	$\longrightarrow$	1
if dot product is	1	$\longrightarrow$	0

So in a sense, $\|\vec{v} \times \vec{w}\|$ measures "how perpendicular" vectors are, and $\vec{v} \cdot \vec{w}$ measures "how parallel" vectors are.

Sec 1.2, 1.3, 1.5 Equations of lines and planes



Given: a point $P_0 = (x_0, y_0, z_0)$ on the line l and a vector $\vec{v}$ parallel to l, say $\vec{v} = \langle a, b, c \rangle$.

Let $P = (x, y, z)$ be any point on l

We see that $\vec{r} = \vec{r}_0 + \vec{a}$.

$\vec{a} \parallel \vec{v}$, so $\vec{a} = t\vec{v}$ for some t. $\longrightarrow$

- Equation of line (vector form): (through P_0 in the direction of $\vec{v}$)

$\boxed{\vec{r} = r_0 + t\vec{v}}$, where $\vec{r}_0$ is ~~parallel~~ the position vector for P_0.

- From $\langle x, y, z\rangle = \vec{r} = \vec{r}_0 + t\vec{v} = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle$ we have parametric form of the equation of a line:

$$x = x_0 + ta, \quad y = y_0 + tb, \quad z = z_0 + tc.$$

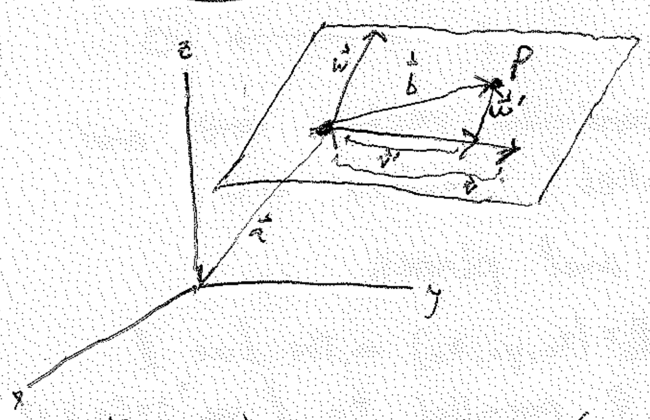
- Solving for t, we have the symmetric equations of a line:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \quad (\text{for } a, b, c \neq 0)$$

Example: Equation of line passing through $(3, 2, -2)$ in the direction of $\vec{i} - \vec{j} + 2\vec{k}$

Soln
1. $\vec{r} = r_0 + t\vec{v} = \langle 3, 2, -2\rangle + t\langle 1, -1, 2\rangle$
2. $x = 3 + t, \; y = 2 - t, \; z = -2 + 2t$
3. $\dfrac{x-3}{1} = \dfrac{y-2}{-1} = \dfrac{z+2}{2}$

- Planes



Plane spanned by $\vec{v}$ and $\vec{w}$

Pick a point $P = (x, y, z)$ on the plane. We can write position vector for P by scaling $\vec{v}$ and $\vec{w}$:

$\vec{p} = \vec{a} + \vec{b}$, where $\vec{b} = \vec{v}' + \vec{w}'$ and $\vec{v}' = t\vec{v}$, $\vec{w}' = s\vec{w}$ $\rightarrow$

Parametric equation of plane spanned by $\vec{v}$ and $\vec{w}$:

$\boxed{\vec{p} = a + t\vec{v} + s\vec{w}}$

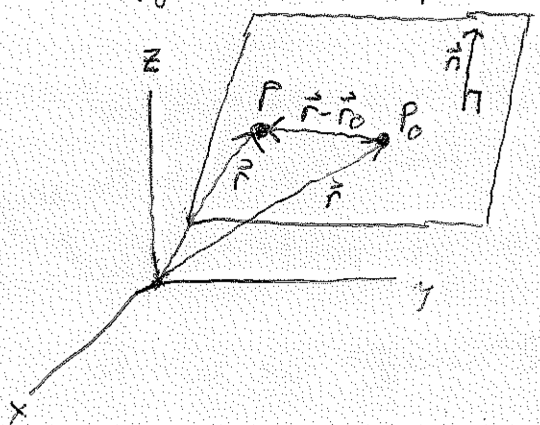
- A vector $\vec{n}$ perpendicular to a plane is called a <u>normal vector</u> to the plane.

- <u>Vector equation of a plane</u>: $\boxed{\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0}$ where

$\vec{n}$ = a normal vector to the plane

$\vec{r} = \langle x, y, z \rangle$ is the position vector of any point on the plane

$\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$ is the position vector of a given point on the plane.



← $\vec{n}$ and $\vec{r} - \vec{r}_0$ are orthogonal, so $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$.

- Compute this dot product to get the <u>scalar equation of a plane</u>:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

or

$$ax + by + cz + d = 0, \text{ where } d = -ax_0 - by_0 - cz_0.$$

- <u>Note</u>: We can read off a normal vector $\vec{n} = \langle a, b, c \rangle$ from any of these forms.

<u>Example</u> Find the equation of the plane containing the points $A = (1, -2, 0)$, $B = (3, 1, 4)$, $C = (0, -1, 2)$.

<u>Soln</u> Need a point and a normal vector. Two vectors in the plane are $\vec{AB} = \langle 2, 3, 4 \rangle$, $\vec{AC} = \langle -1, 1, 2 \rangle$, so $\vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 4 \\ -1 & 1 & 2 \end{vmatrix} = \langle 2, -8, 5 \rangle$.

So an equation of the plane is, using point A,

$$\underbrace{\langle 2, -8, 5 \rangle}_{\vec{n}} \cdot \underbrace{\langle x - 1, y + 2, z - 0 \rangle}_{\vec{r} - \vec{r}_0} = 0$$

(or $2(x-1) - 8(y+2) + 5z = 0$

or $2x - 8y + 5z - 18 = 0$)